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The decay of a plane shock wave

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An analytic solution of the non-isentropic equations of gas-dynamics, for the one-dimensional motion of a non-viscous and non-conductive medium, is derived in this paper for the first time. This is a particular solution which contains only one arbitrary function. On the basis of this solution, the interaction of a centred simple wave with a shock of moderate strength is analyzed; and it is shown that, for a weak shock, this analysis is compatible with Friedrichs's theory. Furthermore, in the light of this analysis, it is explained why the empirical methods employed by the shock-expansion theory, including Whitham's rule for determining the shock path, work.

1. Introduction

During the past two decades, apart from the development of numerical methods, the literature on the decay of shock waves has primarily been concerned either with the treatment of strong shocks by means of the similarity solutions of the equations of gas-dynamics, or with the study of those changes in the speed and the strength of the shock which are entirely caused by the non-uniformity of the region in which the shock propagates. This paper, however, deals with a problem which has received very little attention since Friedrichs's theory: the decay of a plane shock wave as a result of its interaction with a simple wave. The simplest case of a motion involving such an interaction is that caused by the deceleration of a piston which is initially moving with a constant velocity in contact with the constant state behind a uniform shock.

In general, interaction of a simple wave with a shock discontinuity gives rise to two reflected waves: a non-isentropic flow and an isentropic general wave† (see figure 1). That, on the one hand, the boundary of a non-isentropic region carrying a constant value of the entropy travels with the fluid velocity, $dx/dt = u$; and, on the other hand, the boundary of the entire reflected flow has to propagate back into the incident simple wave at the speed of sound relative to the fluid, $dx/dt = u-c$, would imply that upon reflection, in addition to the non-isentropic flow immediately adjacent to the decaying shock, there is also a new isentropic flow created, which has to be described by the general solution of the isentropic equations of motion. Since such a solution is known (see, for example, Courant &

† In this paper, the flow which results from the interaction of two simple waves and is described by the general solution of the isentropic equations of motion is called a general wave.

Friedrichs 1948), this isentropic flow can be uniquely specified by solving the Cauchy's problem at the sonic discontinuity between regions I and II. However, in specifying the non-isentropic flow in region III, we are faced with an initial boundary-value problem with an unknown boundary, whose explicit solution in the general case of a shock of arbitrary strength is not feasible.

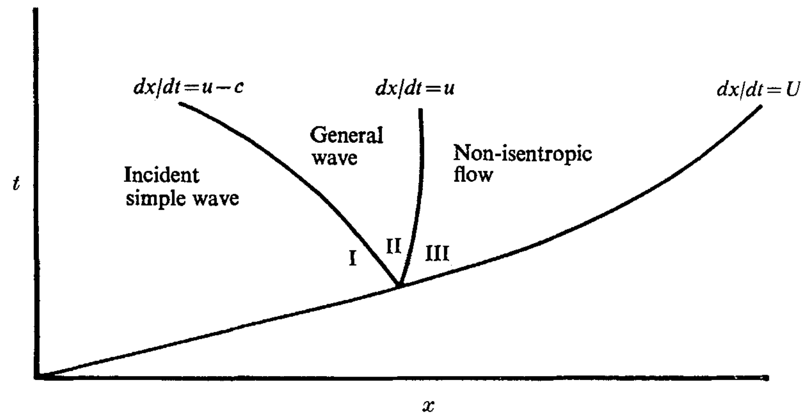

FIGURE 1. Regions III, II and I, whose right-hand boundaries travel at the shock speed U, the particle velocity u, and the speed of sound c relative to the fluid, stand for the non-isentropic flow, the general wave, and the incident simple wave, respectively.

Friedrichs's theory (cf. Friedrichs 1947) supplies a solution to this problem if the initial shock is weak enough for the change of entropy across it to be negligible, i.e. if the flow behind the decaying shock can be assumed to remain isentropic. In this case, the isentropic simple wave solution happens to satisfy the Rankine–Hugoniot conditions at the shock front, and the shock path can be determined by the imposition of the kinematic condition $dx/dt = U$. Within its range of validity, in that the method it employs is duly justified, this is a rigorous theory. There are, however, other methods, generally employed for solving this problem in the case of stronger shock waves (cf. Eggars, Savin & Syvertson 1955; Whitham 1958), which are developed in an *ad hoc* manner; and their results, though remarkably accurate, are apparently quite fortuitous. These methods—which belong to the so-called shock-expansion theory—will be discussed in § 5.

To obtain an explicit solution to the initial boundary-value problem posed here, without limiting the strength of the shock, we need to know the general solution of the non-isentropic equations of motion; this is in order that the two boundary conditions at the shock front and a third condition (which depends on the particular type of the incident simple wave) at the particle path between regions III and II can be satisfied by specifying the three arbitrary functions contained in this general solution. However, since the particular solution to be derived in this paper contains only one arbitrary function, and hence is clearly insufficient to enable us to tackle this problem in its full generality, we are here led to ask the following question: can this solution describe, at least, a specific type of non-isentropic flow—one which arises from the interaction of a specific type of incident simple wave—adjacent to a decaying shock? Although to describe even such a specific type of flow in an exact manner, we need a solution more general than the one derived here, the analysis which follows will show that this particular solution can in fact describe one such flow to within a reasonable degree of approximation. If the single arbitrary function contained in this solution is determined such that the kinematic boundary condition, $dx/dt = U$, is satisfied exactly, then it is possible to satisfy the Rankine–Hugoniot conditions approximately by setting a limit on the strength of the shock; a limit, however, which does not have to be so restrictive as in the case of Friedrichs's theory. It turns out that the corresponding incident simple wave, whose interaction with a plane shock will give rise to this specific type of non-isentropic flow, is a centred simple wave, i.e. the type of simple wave which is created by a sudden halt or recession in the motion of the piston.

The solution presented here, besides being applicable to stronger shocks, describes certain non-isentropic features of the decay of a shock wave whose analysis is beyond the scope of Friedrichs's theory; although in another respect, in that it only deals with the case of a centred simple wave, it is more limited in its scope than the latter. The main contribution of this solution, however, lies in its throwing light on some of the empirical methods applied to this problem in the past.

2. Derivation of a particular solution of the non-isentropic equations of motion

The Eulerian equations governing the one-dimensional motion of a fluid in the absence of dissipative forces are

$$\left.\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)+\frac{\partial p}{\partial x} &= 0,\\ \frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x} &= 0,\\ \frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x} &= 0, \end{aligned}\right\} \tag{2.1}$$

where u stands for the fluid velocity, and p, ρ and S are the pressure, the density and the entropy of the fluid, respectively. Written in terms of the speed of sound $c = [(\partial p/\partial \rho)_S]^{\frac{1}{2}}$ by means of the equation of state for an ideal gas, these equations assume the alternative form

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{2}{\gamma-1}c\frac{\partial c}{\partial x}-\frac{c^2}{\gamma(\gamma-1)c_v}\frac{\partial S}{\partial x} &= 0,\\ \frac{\partial c}{\partial t}+u\frac{\partial c}{\partial u}+\frac{\gamma-1}{2}c\frac{\partial u}{\partial x} &= 0,\\ \frac{\partial S}{\partial t}+u\frac{\partial S}{\partial x} &= 0, \end{aligned}\right\} \tag{2.2}$$

where γ is the ratio of the specific heats c_p and c_v.

To put these equations in a form more suited to a hodograph transformation, let us introduce the following new independent variables

$$\xi = \ln\left[c^{2/(\gamma-1)}\exp\left(-\frac{1}{\gamma(\gamma-1)}\frac{S-S_0}{c_v}\right)\right], \quad \eta = \ln c^{2/(\gamma-1)}, \tag{2.3}$$

to obtain

$$\left.\begin{aligned} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+e^{(\gamma-1)\eta}\frac{\partial \xi}{\partial x} &= 0,\\ \frac{\partial \xi}{\partial t}+u\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial x} &= 0,\\ \frac{\partial \eta}{\partial t}+u\frac{\partial \eta}{\partial x}+\frac{\partial u}{\partial x} &= 0;\end{aligned}\right\} \tag{2.4}$$

which in terms of Jacobians become

$$\begin{aligned} \frac{\partial(u,x)}{\partial(t,x)}+u\frac{\partial(t,u)}{\partial(t,x)}+e^{(\gamma-1)\eta}\frac{\partial(t,\xi)}{\partial(t,x)} &= 0,\\ \frac{\partial(\xi,x)}{\partial(t,x)}+u\frac{\partial(t,\xi)}{\partial(t,x)}+\frac{\partial(t,u)}{\partial(t,x)} &= 0,\\ \frac{\partial(\eta,x)}{\partial(t,x)}+u\frac{\partial(t,\eta)}{\partial(t,x)}+\frac{\partial(t,u)}{\partial(t,x)} &= 0.\end{aligned}$$

Now, the fact that for a non-isentropic flow ξ and η are independent enables us to interchange the roles of the dependent and the independent variables in these equations by multiplying every one of them by the non-zero Jacobian $\partial(t,x)/\partial(\xi,\eta)$. The resulting equations, when written out in terms of derivatives, yield

$$\left.\begin{aligned} \frac{\partial u}{\partial \xi}\frac{\partial x}{\partial \eta}-\frac{\partial u}{\partial \eta}\frac{\partial x}{\partial \xi}+u\left(\frac{\partial u}{\partial \eta}\frac{\partial t}{\partial \xi}-\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \eta}\right)-e^{(\gamma-1)\eta}\frac{\partial t}{\partial \eta} &= 0,\\ \frac{\partial x}{\partial \eta}-u\frac{\partial t}{\partial \eta}+\frac{\partial u}{\partial \eta}\frac{\partial t}{\partial \xi}-\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \eta} &= 0,\\ \frac{\partial x}{\partial \xi}-u\frac{\partial t}{\partial \xi}-\frac{\partial u}{\partial \eta}\frac{\partial t}{\partial \xi}+\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \eta} &= 0.\end{aligned}\right\} \tag{2.5}$$

Let us next denote the recurring expression in these equations by the function

$$\psi(\xi,\eta) = \frac{\partial u}{\partial \eta}\frac{\partial t}{\partial \xi}-\frac{\partial u}{\partial \xi}\frac{\partial t}{\partial \eta}, \tag{2.6}$$

and insert the partial derivatives of x from the second and the third of the above equations in the first equation, to obtain

$$\left.\begin{aligned} e^{(\gamma-1)\eta}\frac{\partial t}{\partial \eta}+\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right)\psi &= 0,\\ \frac{\partial x}{\partial \eta} &= u\frac{\partial t}{\partial \eta}-\psi,\\ \frac{\partial x}{\partial \xi} &= u\frac{\partial t}{\partial \xi}+\psi.\end{aligned}\right\} \tag{2.7}$$

Since we must have $\partial^2 x/\partial\xi\,\partial\eta = \partial^2 x/\partial\eta\,\partial\xi$, however, the auxiliary function ψ must satisfy the following differential equation

$$\frac{\partial\psi}{\partial\xi}+\frac{\partial\psi}{\partial\eta}+\psi = 0,$$

whose integral in terms of an arbitrary function of $\xi-\eta$ is

$$\psi = e^{-\eta}g(\xi-\eta). \tag{2.8}$$

As long as (2.8) is satisfied, the second and third equations in (2.7) can be looked upon as the definition of x in terms of u and t. This therefore implies that the system of equations (2.7) is, in effect, reduced to the following two equations:

$$\frac{\partial u}{\partial\eta}\frac{\partial t}{\partial\xi}-\frac{\partial u}{\partial\xi}\frac{\partial t}{\partial\eta}-e^{-\eta}g(\xi-\eta) = 0, \tag{2.9}$$

and

$$e^{(\gamma-1)\eta}\frac{\partial t}{\partial\eta}+\left(\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}\right)e^{-\eta}g(\xi-\eta) = 0, \tag{2.10}$$

which are obtained by substituting (2.8) in the definition of ψ and in the first equation of (2.7), respectively. If we now solve these two equations for $\partial t/\partial\xi$ and $\partial t/\partial\eta$ and require that $\partial^2 t/\partial\xi\,\partial\eta = \partial^2 t/\partial\eta\,\partial\xi$, the following single differential equation for $u = u(\xi,\eta)$ results

$$\frac{\partial}{\partial\eta}\left[e^{-\gamma\eta}g\left\{\frac{\dfrac{\partial u}{\partial\xi}\left(\dfrac{\partial u}{\partial\xi}+\dfrac{\partial u}{\partial\eta}\right)-e^{(\gamma-1)\eta}}{\partial u/\partial\eta}\right\}\right] = \frac{\partial}{\partial\xi}\left[e^{-\gamma\eta}g\left(\frac{\partial u}{\partial\xi}+\frac{\partial u}{\partial\eta}\right)\right]. \tag{2.11}$$

This is a second-order partial differential equation which will later be seen to be of the hyperbolic type as expected.

At this point, let us make a conjecture. Noting that $\xi-\eta$ is a function of entropy only, and that in the isentropic limit $u = u(\xi,\eta)$ should reduce to the Riemann invariant

$$u = \frac{2}{\gamma-1}e^{\frac{1}{2}(\gamma-1)\eta}+\text{const.},$$

let us venture to try the following for a solution of (2.11)

$$u = \frac{2}{\gamma-1}e^{\frac{1}{2}(\gamma-1)\eta}h(\xi-\eta)+\text{const.}, \tag{2.12}$$

where h is an arbitrary function of $\xi-\eta$. In fact, when (2.12) is substituted in the differential equation (2.11), it so happens that all the exponential terms, which are function of η only, disappear and the resulting equation assumes the form of a differential relation between the two functions $g(\xi-\eta)$ and $h(\xi-\eta)$. That is to say, the expression (2.13) does, in fact, satisfy the differential equation (2.11) as long as one of the two functions g and h is expressed in terms of the other by the resulting differential relation. If we choose to express g in terms of h, then the expressions obtained earlier for the partial derivatives of x and t can also be expressed in terms of h and integrated to yield a solution of the system of equations (2.5) in terms of this arbitrary function.

In the following representation of this solution, the function f has been introduced instead of a certain combination of g and h, and ξ and η have been replaced by the original variables c and S:

$$\left.\begin{aligned} x &= [u+ch(\sigma)]t, \\ t &= c^{-(\gamma+1)/(\gamma-1)}f(\sigma), \\ f &= (h^2-1)^{-(\gamma+1)/2(\gamma-1)}\exp\left[\frac{\gamma+1}{2}\int\frac{d\sigma}{h^2-1}\right], \\ u &= \frac{2}{\gamma-1}ch(\sigma)+\text{const.}, \end{aligned}\right\} \tag{2.13}$$

where the new variable σ is defined by

$$\sigma = -\frac{1}{\gamma(\gamma-1)}\frac{S-S_0}{c_v}.$$

It should be noted that solution (2.13) is not directly reducible to an isentropic solution by setting $\sigma = \text{const.}$ in the above equations, since, in deriving this solution, it was assumed that ξ and η were independent.

3. The non-isentropic flow

A solution describing the non-isentropic flow just behind a decaying shock has to satisfy two boundary conditions along the shock path: the Rankine–Hugoniot relations and the kinematic condition $dx/dt = U$. According to whether the arbitrary function h contained in solution (2.13) is specified by imposing one or the other of these two boundary conditions, therefore, we obtain two different solutions. In this section, after having thus specified the two alternative forms of this arbitrary function, we shall attempt to find out whether there are any limits within which the difference between the corresponding values of these two functions is negligibly small.

In a frame of reference in which the gas in front of the shock is stationary, the Rankine–Hugoniot relations for an ideal gas can be written as

$$\left.\begin{aligned} \frac{\rho}{\rho_0} = \left[\frac{\gamma+1}{2\gamma}\pi+1\right]\Big/\left[\frac{\gamma-1}{2\gamma}\pi+1\right], \quad \frac{u}{c_0} = \frac{1}{\gamma}\pi\Big/\left[\frac{\gamma+1}{2\gamma}\pi+1\right]^{\frac{1}{2}}, \\ \frac{U-u}{c_0} = \left[\frac{\gamma-1}{2\gamma}\pi+1\right]\Big/\left[\frac{\gamma+1}{2\gamma}\pi+1\right]^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.1}$$

where the variable π is defined by

$$\pi = \frac{p}{p_0}-1,$$

and the constant values of the flow variables within the stationary gas ahead of the shock are designated by the subscript zero. For such a gas, the entropy and the speed of sound $c = [\gamma p/\rho]^{\frac{1}{2}}$ can also be expressed as functions of π at the shock front with the aid of the first relation in (3.1) and the equation of state

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}\exp\left[\frac{S-S_0}{c_v}\right]. \tag{3.2}$$

Hence, to impose the kinematic condition at the shock front, let us differentiate the expressions given by solution (2.13) for x and t with respect to π, and insert the results in $dx/dt = U$, to arrive at the following ordinary differential equation:

$$\frac{dh}{d\pi} = \left[-\frac{2}{\gamma-1}\frac{d\ln c}{d\pi}h^3 + \left(\frac{1}{c}\frac{du}{d\pi} + \frac{\gamma+1}{\gamma-1}\frac{U-u}{c}\frac{d\ln c}{d\pi}\right)h^2 \right.$$
$$+\left(\frac{\gamma+1}{2}\frac{d\sigma}{d\pi} + \frac{2}{\gamma-1}\frac{d\ln c}{d\pi}\right)h - \frac{\gamma+1}{2}\frac{U-u}{c}\frac{d\sigma}{d\pi} - \frac{\gamma+1}{\gamma-1}\frac{U-u}{c}\frac{d\ln c}{d\pi}$$
$$\left. -\frac{1}{c}\frac{du}{d\pi}\right] \Big/ \left[\frac{2}{\gamma-1}h^2 - \frac{\gamma+1}{\gamma-1}\frac{U-u}{c}h + 1\right], \tag{3.3}$$

in which the flow variables U, u, c and σ are to be expressed in terms of π. If we now integrate this differential equation using the initial conditions $h = 1$ when $\pi = 0$, we will obtain the reduced form of the function h along the shock path: $h = h(\pi)$. (The choice of these initial conditions is dictated by the fact that, in the limiting case of a sonic discontinuity, the last relation in solution (2.13) must reduce to a Riemann invariant.) The original form of this function, $h = h(\sigma)$, can then be retrieved by means of the relationship which holds between π and σ at the shock front. This relationship and the numerical results of the above integration for $\gamma = \frac{5}{3}$ are shown in table 1.

π	$h(\pi)$	$\exp\left[\frac{S-S_0}{c_v}\right]$
0	1·0000	1·0000
1	1·0028	1·0175
2	1·0096	1·0692
3	1·0168	1·1388
4	1·0233	1·2181
5	1·0290	1·3029

TABLE 1

Next, in order to determine the alternative form of the arbitrary function h which will, on the other hand, satisfy the Rankine–Hugoniot conditions exactly, let us consider the last relation in solution (2.13). If the constant in this relation is determined by once again requiring that for a sonic discontinuity ($\pi = 0$), propagating into a region of quiet gas ($u = 0$, $c = c_0$), h must have the value 1, and solve the resulting equation for h, we will arrive at

$$h = 1 + \frac{u - [2/(\gamma-1)](c-c_0)}{[2/(\gamma-1)]c}. \tag{3.4}$$

When the variables u and c in it are expressed in terms of π, this equation will yield $h = h(\pi)$ at the shock front. Note, however, that the numerator of the second term in this equation represents the change of one of the Riemann invariants through a weak shock; and hence, in the limit of $\pi \ll 1$, has a vanishingly small value (see Courant & Friedrichs 1948). Therefore, comparing the values of h obtained above with 1, i.e. with the value given by equation (3.4) in the case of a weak shock, we can readily see that, as long as $\pi \ll 1$, the two alternative forms of this function tally.

The question to be asked now is: up to what shock strengths would the corresponding values of $h(\pi)$ given by equations (3.3) and (3.4) remain close to one another? It should be noted that, unlike in the case of Friedrichs's theory, here the strength of the shock is not limited by the range of validity of the solution, but by the degree of approximation with which both of the boundary conditions at the shock front can be satisfied. For this reason, this compatibility requirement turns out not be to so restrictive as to exclude those shocks through which the

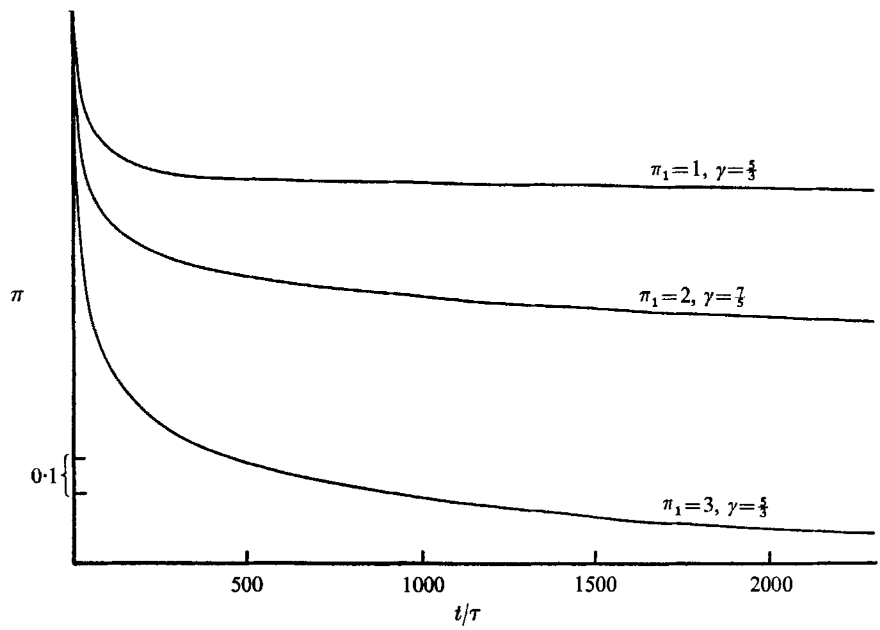

FIGURE 2. Variation of pressure at the shock front. Here, each curve is plotted starting at the same point on the π axis; where they actually start from is implied by the initial strength of the shock π_1. The errors involved are 5 %, 3 % and 0·7 % for $\pi_1 = 3$, 2, and 1, respectively.

change of entropy is appreciable. In fact, in the case of $\gamma = \frac{5}{3}$, the difference between the corresponding values of the two alternative forms of h is less than 0·05, even for shocks as strong as $\pi = 3$. If this discrepancy is taken to be a measure of our approximation, then this implies that for a decaying shock which starts with the initial strength of $\pi_1 = 3$, solution (2.13) satisfies both of the boundary conditions to within a 5 % error. To within a 5 % error, however, the change of entropy across a shock of this strength is not zero. In the case of $\gamma = \frac{5}{3}$, for the change of entropy across the shock to be zero to within a 5 % error, the shock should start with an initial strength of $\pi_1 \leqslant 1{\cdot}6$.

In plotting the following graphs, illustrating certain features of the decay of shocks of different initial strengths, for the two values of the adiabatic index $\gamma = \frac{5}{3}$ and $\gamma = \frac{7}{5}$, we have used the expression for $h(\sigma)$ which is obtained by integrating equation (3.3), and have, in each case, indicated the degree of approximation involved. The unit of time in these calculations is chosen to be the time it takes the sonic discontinuity in front of the incident simple wave to travel a unit of distance towards the initial uniform shock. That is, if the characteristic time and the characteristic distance of the problem are denoted by τ and λ

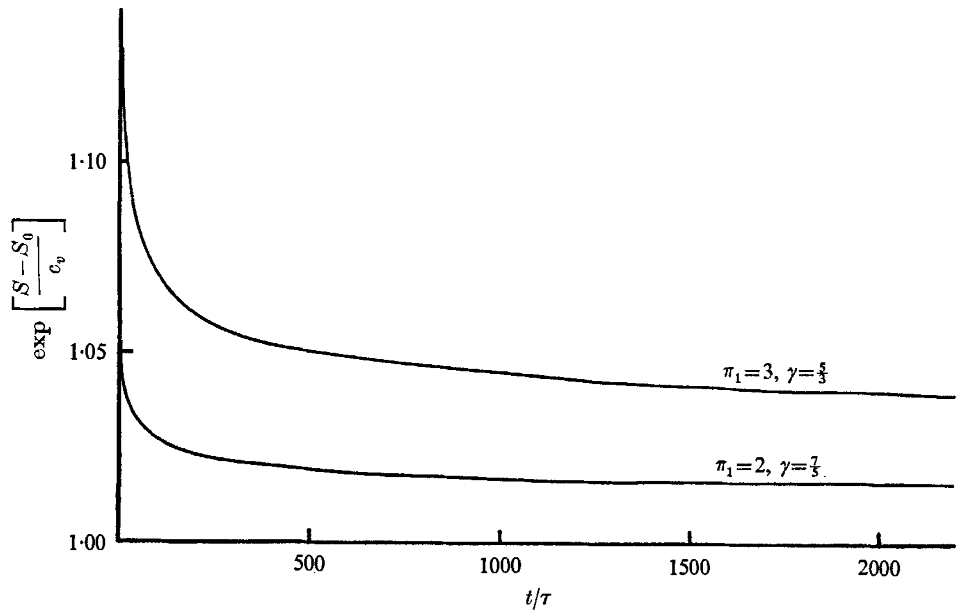

FIGURE 3. Variation of entropy at the shock front.

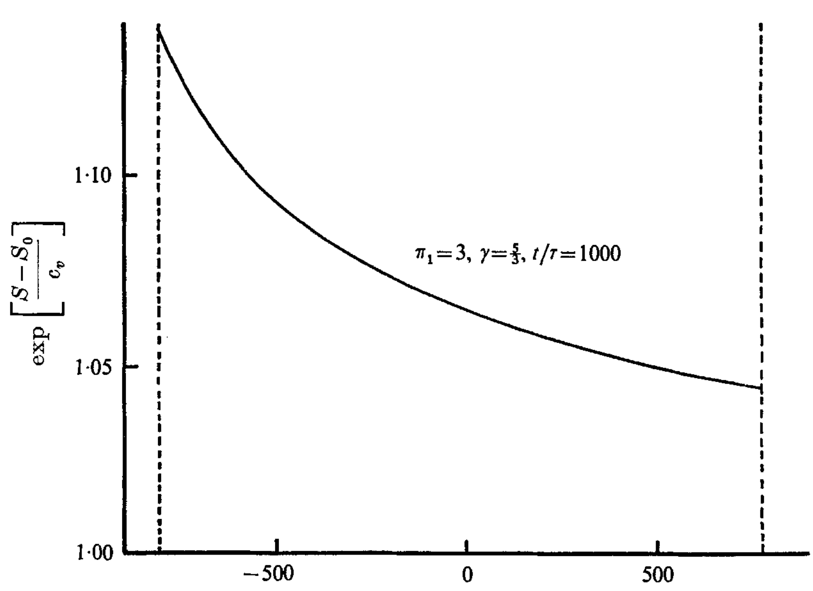

FIGURE 4. Distribution of entropy within the non-isentropic flow at a given time.

respectively, then $\lambda = (u_1 + c_1)\tau$, where the subscript 1 designates the value of a flow variable within the constant state behind the initial shock. It is further assumed that the incident simple wave arrives at the shock front at $t/\tau = 1$, $x/\lambda = 1$.

Since the values of $h(\sigma)$ depart from 1 only by a small amount, it can readily be seen from the first equation in solution (2.13) that, to within the degree of our approximation, distributions of the fluid velocity u and the speed of sound c

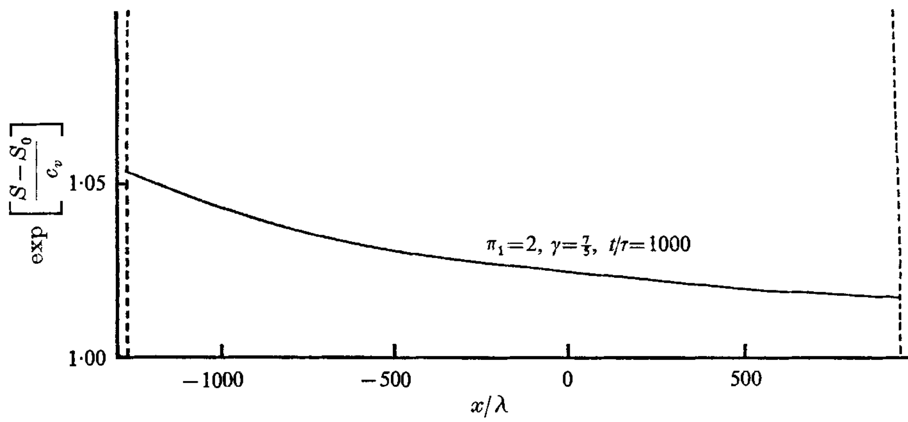

FIGURE 5. Distribution of entropy within the non-isentropic flow at a given time.

in the region immediately behind the shock are linear. Friedrichs's theory will yield this same result if the incident flow is a centred simple wave which, as we shall see in the following section, is the case here. That the law of decay of the shock front calculated by means of the non-isentropic solution (2.13) behaves like $t^{-\frac{1}{2}}$ as $t \to \infty$, i.e. that in the limit of a very weak shock our analysis also yields the same result as that of Friedrichs's theory, is demonstrated in the appendix.

4. The incident simple wave

Having determined the specific non-isentropic flow—which to within a certain degree of approximation can be described by our particular solution—in region III, we shall now proceed to specify the corresponding flows in regions II and I by solving Cauchy's problem twice at the two relevant boundaries of these regions. The general solution of the isentropic equations of motion is known for any value of λ in terms of hypergeometric functions. However, as will become clear later in this section, the choice of the following form of this solution which corresponds to the value of $\gamma = \frac{5}{3}$ (cf. Stanyukovich 1960), for our present purpose, will result in no loss of generality:

$$\left.\begin{aligned} x &= ut - \frac{\partial \psi}{\partial u}, \quad t = \frac{\partial \psi}{\partial i}, \\ \psi &= i^{-\frac{1}{2}}\{F_1[(6i)^{\frac{1}{2}} + u] + F_2[(6i)^{\frac{1}{2}} - u]\}, \end{aligned}\right\} \tag{4.1}$$

where $i = c^2/(\gamma - 1)$ is the enthalpy, and F_1 and F_2 are two arbitrary functions.

If, in order to satisfy the two boundary conditions at the weak discontinuity between regions III and II, we require that at this boundary the expressions for x and t given by the two solutions (2.13) and (4.1) in the case of $\gamma = \frac{5}{3}$ match, we will arrive at the following two equations:

$$\frac{\partial\psi}{\partial u}+h_1 c\frac{\partial\psi}{\partial i} = 0, \quad \frac{\partial\psi}{\partial i}-f_1 c^{-4} = 0, \tag{4.2}$$

in which h_1 and f_1 stand for $h(\sigma_1)$ and $f(\sigma_1)$ respectively; since the entropy at this boundary, σ_1, is constant, so are h_1 and f_1. Note, however, that at this boundary the two variables u and i are not independent: the last equation in solution (2.13) yields

$$u = h_1(6i)^{\frac{1}{2}}+K, \tag{4.3}$$

where $K = -3c_0$, as calculated in the preceding section. Hence, after having rewritten the partial derivations of ψ in the above two equations in terms of F_1, F_2, and the total derivatives of F_1 and F_2 with respect to their arguments, it is possible to transform equations (4.2) into two ordinary differential equations by employing the relationship (4.3) to express $(6i)^{\frac{1}{2}}+u$ and $(6i)^{\frac{1}{2}}-u$ in terms of a single variable. Integrating the resulting differential equations for F_1 and F_2, and making a further use of the relationship (4.3) to retrieve the original arguments of these two functions, we will obtain

$$\left.\begin{aligned} F_1[(6i)^{\frac{1}{2}}+u] &= \alpha[(6i)^{\frac{1}{2}}+u-K]^{-1}, \\ F_2[(6i)^{\frac{1}{2}}-u] &= \alpha[(6i)^{\frac{1}{2}}-u+K]^{-1}, \end{aligned}\right\} \tag{4.4}$$

where

$$\alpha = -\frac{9(6)^{\frac{1}{2}}}{8}f_1(h_1^2-1)^2. \tag{4.5}$$

Next, insertion of the above two expressions in solution (4.1) yields an exact description of the flow in region II. Before discussing the nature of this flow, however, let us specify the incident simple wave in region I.

Since this simple wave is initially adjacent to the constant state, $u = u_1$, $c = c_1$, behind the original uniform shock, it can be represented by

$$x = (u+c)t+g(u), \tag{4.6}$$

$$u = 3c+K_0, \tag{4.7}$$

where $K_0 = u_1-3c_1$, and the arbitrary function $g(u)$ is given by

$$-g(u) = \frac{\partial\psi}{\partial u}+c\frac{\partial\psi}{\partial i};$$

an equation which is obtained by requiring that the general and the simple wave solutions match across the sonic discontinuity between regions II and I. Inserting the expressions given by equations (4.4) for F_1 and F_2 into this equation, and making use of the relationship (4.7), we obtain

$$g(u) = 6^{\frac{1}{2}}\alpha[2(u-K_0)(2u-K_0-K)^{-2}+(2u-K_0-K)^{-1}+(K-K_0)^{-1}]. \tag{4.8}$$

The above descriptions of the flows in regions I and II have so far been obtained by making the implicit assumption that the description of the flow in region III is exact. In order to be consistent, however, we should also subject these results to the approximation used earlier in the analysis of the flow in region III. Equation (4.5) clearly indicates that to within the degree of approximation employed in satisfying the Rankine–Hugoniot conditions at the shock front, i.e. in the limit of $h_1 - 1 \ll 1$, the constant α equals zero. Since the first two terms inside the brackets in equation (4.8) are finite and the third term is of the order of $(h_1 - 1)^{-1}$, it then follows that in this limit $g(u)$ is also zero. That is to say, the particular simple wave whose interaction with a shock discontinuity gives rise to the particular non-isentropic flow considered in § 3 is, in fact, a centred simple wave.

Furthermore, in this limit, the departure of the isentropic general wave of region II from a centred simple wave also turns out to be negligibly small. To demonstrate this, let us first divide the expressions given by equations (4.1) and (4.4) for $x = x(u, i)$ and $t = t(u, i)$ to arrive at

$$x/t = u + \tfrac{1}{3}(u - K), \tag{4.9}$$

a result which is independent of the constant α and yields u as a function of x and t. If we next insert $u = u(x, t)$ in $t = t(u, i)$ and solve the resulting equation for $i = i(x, t)$, the expressions obtained for u and i as functions of x and t can then be employed to show that one of the Riemann invariants within this general wave has the following dependence on x and t

$$u - (6i)^{\frac{1}{2}} = \frac{1}{4}\left(3\frac{x}{t} + K\right)(1 - \phi) + K\phi,$$

where

$$\phi(x, t) = 1 + [16(h_1^2 - 1) f_1^{\frac{1}{2}}] \Big/ \left[\left(\frac{x}{t} - K\right)^2 t^{\frac{1}{2}}\right].$$

Since f/t remains finite throughout the motion, however, this equation yields

$$\lim_{h_1 \to 1} [u - (6i)^{\frac{1}{2}}] = K. \tag{4.10}$$

Not only does the constancy of the above Riemann invariant imply that the flow in region II has to be a simple wave in the limit, but also (4.10) inserted in (4.9) yields the expression $x/t = u + c$, which explicitly specifies the type of this simple wave as a centred one.

5. Remarks on shock-expansion theory

Shock-expansion theory, which is generally employed in calculating the pressure distribution on an aerofoil in a uniform supersonic stream, deals with the mathematical analogue of the problem considered here in the case of a two-dimensional steady flow. Although its application is not restricted to weak shocks, this theory differs from Friedrichs's theory only in that it adopts a different value of the Riemann invariant, i.e.

$$u - \frac{2}{\gamma - 1} c = u_1 - \frac{2}{\gamma - 1} c_1 \tag{5.1}$$

instead of $2c_0/(\gamma-1)$. However, that the distribution of pressure calculated by means of the above formula corresponds remarkably well to the experimental results for a wide range of shock strengths (Eggars *et al.* 1955) is quite fortuitous; not only is the flow behind a non-uniform shock assumed to be isentropic even when the shock is not weak, but also the choice of the constant in the above formula is made without any justification.

Mahony (1955) attempts to justify this theory by means of a numerical calculation in which the first-order changes of entropy across the shock are taken into account. His conclusion is that although the errors arising from using an isentropic solution, on the one hand, and making an incorrect choice of the value of the Riemann invariant, on the other hand, 'cancel out almost exactly in the examples of circular-arc aerofoils'; nevertheless, 'the accuracy of shock-expansion theory in the extreme cases, ..., is to some extent a matter of chance'.

Relevant to shock-expansion theory, in that formula (5.1) can be obtained by means of it, is furthermore the following simple rule due to Whitham (1958). If the non-isentropic equations of motion are first written in a characteristic form, then this rule consists of applying the differential relation which must be satisfied by the flow variables along a characteristic, in conjunction with the Rankine–Hugoniot relations, to the flow variables just behind the shock front to determine the shock path. In connexion with this rule also—since in no case, other than that of a shock whose strength has reduced to zero, is the shock path a characteristic of the equations of motion—as Whitham remarks, 'the accuracy of the results for a wide range of problems and for all shock strengths is truely surprising'.

Surprising these results certainly are; but here at least, for shocks of moderate strength, is the reason why shock-expansion theory works: there happens to be a striking resemblence between the non-isentropic flow just behind a decaying shock and an isentropic simple wave. This is readily seen from the first and the last equations in solution (2.13), and the values of h given in table 1. In fact, formula (5.1), employed in shock-expansion theory, can be directly derived from the last equation in this solution by imposing the relevant initial conditions behind the original uniform shock: when $u = u_1$ and $c = c_1$, $h = h_1 \sim 1$.

In a similar way, Whitham's rule works because the differential relation used in this rule happens to be the same as that given by solution (2.13). Since the differential relation to be applied to the flow variables just behind the shock front should be obtained from a solution of the equations of motion, and not from their characteristic form, the correct relation is

$$du = \frac{2}{\gamma-1}hdc - \frac{2c}{\gamma(\gamma-1)^2}\frac{dh}{d\sigma}\frac{dS}{c_v},$$

which is obtained by differentiating the last equation in solution (2.13). However, it turns out that letting $h \sim 1$ in equation (3.3), we get $dh/d\sigma \sim (\gamma-1)/2$, which inserted in the above equation will yield a differential relation precisely the same as that which must be satisfied by the flow variables along a characteristic of the equations of motion.

6. Concluding remarks

It should be pointed out that the hodograph co-ordinates ξ and η, introduced in this paper, may in fact prove useful even in obtaining numerical solutions of the non-isentropic equations of motion. Apart from reducing the order of these equations, see equation (2.11), the further simplification offered by this transformation lies in the fact that, in the (ξ, η) plane, the shock path is known: the Rankine–Hugoniot conditions together with the equation of state (3.2) yield $\xi = \xi(\pi)$ and $\eta = \eta(\pi)$. Admittedly the arbitrary function $g(\xi-\eta)$ appearing in equation (2.11) introduces new computational problems; however, it can be seen from the following form of this equation

$$A\frac{\partial^2 u}{\partial \xi^2}+2B\frac{\partial^2 u}{\partial \xi\,\partial \eta}+C\frac{\partial^2 u}{\partial \eta^2}+D = 0,$$

$$A = \left(\frac{\partial u}{\partial \eta}\right)^2, \quad B = -\frac{\partial u}{\partial \xi}\frac{\partial u}{\partial \eta}, \quad C = \left(\frac{\partial u}{\partial \xi}\right)^2 - e^{(\gamma-1)\eta},$$

$$D = \frac{\partial u}{\partial \eta}\left\{\gamma\frac{\partial u}{\partial \xi}\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right) - e^{(\gamma-1)\eta} + \frac{g'}{g}\left[\left(\frac{\partial u}{\partial \xi}+\frac{\partial u}{\partial \eta}\right)^2 - e^{(\gamma-1)\eta}\right]\right\},$$

that the arbitrary function $g(\xi-\eta)$ does not appear in the coefficients of the second-order terms which play a much greater part in governing the behaviour of the solution.

A similar transformation can also be applied to the relativistic equations of gas-dynamics to reduce the corresponding relativistic problem to that of solving a second-order partial differential equation for which Cauchy's boundary conditions at the shock front are known (see Ardavan-Rhad 1969).

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Appendix

To show that the analysis presented in this paper is compatible with Friedrichs's theory, we shall here derive the limiting law of motion of a weak shock on the basis of solution (2.13). For this purpose, it is more convenient to consider the following differential form of the third equation in this solution:

$$\frac{d\ln f}{d\sigma} = -\frac{\gamma+1}{2}\frac{[2/(\gamma-1)]\,(dh/d\sigma)\,h-1}{h^2-1}. \tag{A 1}$$

This is because, in the limiting case of $h \to 1$ and $\pi \to 0$, an explicit expression can be found for $dh/d\sigma$ by expanding the numerator and the denominator of equation (3.3) in powers of $h-1$ and π:

$$\frac{dh}{d\sigma} = \frac{\gamma-1}{2}\left[\pi^3 - \frac{4\gamma}{\gamma+1}\left(\frac{8\gamma^2}{\gamma+1}-\pi^2\right)(h-1)\right]\Big/\left[\pi^3+\left(\frac{4\gamma(3-\gamma)}{(\gamma+1)^2}\pi^2+\pi^3\right)(h-1)\right]. \tag{A 2}$$

Note, incidentally, that the result used in §5, i.e. that in the limit of $h \rightarrow 1$, $dh/d\sigma = 2/(\gamma - 1)$, follows from this expression immediately.

To determine the limiting behaviour of the function f, therefore, it suffices to insert the expression given by (A 2) for $dh/d\sigma$ in equation (A 1), replace σ by a third-order term in the shock strength, and let $h \rightarrow 1$; the resulting equation when integrated yields $f \sim \pi^{-2}$. Since, in this limit, $c \sim c_0$, however, using the second equation in solution (2.13), we arrive at $t \sim \pi^{-2}$: a result which is precisely the same as that obtained by means of Friedrichs's theory.

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